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PERFECT MATCHINGS IN BALANCED HYPERGRAPHS

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We generalize Hall's condition for the existence of a perfect matching in a bipartite graph, to balanced hypergraphs.

One of the first results on matchings in graphs is the following celebrated theorem of Hall [6]:

Theorem 1. *A bipartite graph $G(V,E)$ has no perfect matching if and only if there exist disjoint node sets R and B such that $|B|>|R|$ and every edge having one endnode in B has the other in R.*

Berge [1] introduced the following generalization of bipartite graphs: A hypergraph $H(V,E)$ is *balanced* if every odd cycle C of H has an edge containing at least three nodes of C. We refer to Berge [2] for an introduction to the theory of hypergraphs.

Here we prove the following theorem:

Theorem 2. *A balanced hypergraph $H(V,E)$ has no perfect matching if and only if there exist disjoint node sets R and B such that that $|B|>|R|$ and every edge contains at least as many nodes in R as in B.*

It is well known that a bipartite graph with maximum degree Δ contains Δ edge-disjoint matchings. The same property also holds for balanced hypergraphs.

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(Corollary 1 of Theorem 8 in Chapter 5 of [1]). Observations due to Bill Cunningham and Dominique De Werra show that Theorem 2 also implies this property.

Corollary 3. *The edges of a balanced hypergraph H with maximum degree Δ, can be partitioned into Δ matchings.*

Proof. By adding edges containing a unique node, we can assume that H is Δ-regular. (This operation does not destroy the property of being balanced). We now show that H has a perfect matching. Assume not and let R and B be disjoint node sets such that $|R \cap E| \geq |B \cap E|$ for every edge E of H. Adding these inequalities, we get $|R| \geq |B|$ since H is Δ-regular. Since H is balanced, it follows from our theorem that H contains a perfect matching M. Removing the edges of M, the result now follows by induction. ∎

The *node-edge incidence matrix* $A = (a_{ij})$ of a hypergraph $H(V, E)$ is a 0, 1 matrix whose rows are indexed by the nodes in V, whose columns are indexed by the edges in E and whose entries $a_{ij} = 1$ if edge j contains node i, 0 otherwise. A 0, 1 matrix A is *balanced* whenever it is the incidence matrix of a balanced hypergraph H. The proof of Theorem 2 uses integrality properties of some polytopes associated with a balanced 0, 1 $m \times n$ matrix A. Let a^i denote the i^{th} row of A, I the identity matrix and $\mathbf{1}$ the vector of all 1's.

Lemma 4. *The polyhedron $P = \{x, s, t \mid Ax + Is - It = 1,\ x, s, t \geq 0\}$ has integral extreme points when A is a balanced 0, 1 matrix.*

Proof. Let $\bar{x}$, $\bar{s}$, $\bar{t}$ be an extreme point of P. Then $\bar{s}_i \bar{t}_i = 0$ for $i = 1, \ldots, m$, since the corresponding columns are linearly dependent. Let $Q = \{x \mid a^i x \geq 1$, if $\bar{t}_i > 0$, $a^i x \leq 1$, if $\bar{s}_i > 0$, $a^i x = 1$, otherwise, $x \geq 0\}$. Fulkerson, Hoffman and Oppenheim [5] show that Q is an integer polyhedron. Since $\bar{x}$ is an extreme point of Q, $\bar{x}$ is an integral vector and so are $\bar{s}$ and $\bar{t}$. ∎

A system of linear constraints is *totally dual integral* (TDI) if, for each integral objective function vector, the dual linear program has an integral optimal solution (if an optimal solution exists).

Lemma 5. *The linear system, $Ax + Is - It = \mathbf{1}$, x, s, $t \geq 0$ is TDI when A is a balanced 0, 1 matrix.*

Proof. Consider the linear program:

$$\begin{aligned} \max\ & bx + cs + dt \\ & Ax + Is - It = \mathbf{1} \\ & x,\ s,\ t \geq 0 \end{aligned} \tag{1}$$

and its dual:

$$\begin{aligned} \min\ & \mathbf{1}y \\ & yA \geq b \\ & -y \geq d \\ & y \geq c. \end{aligned} \tag{2}$$

Let A be a 0, 1 balanced matrix with smallest number of rows such that the lemma does not hold. Then there exist integral vectors b, c, d, such that an optimal solution of (2), say $\bar{y}$, has a fractional component $\bar{y}_i$. Consider the following linear program:

$$\begin{aligned} &\min \mathbf{1}y \\ &yA^i \geq b - \lceil \bar{y}_i \rceil a^i \\ (3) \qquad &-y \geq d^i \\ &y \geq c^i \end{aligned}$$

where A^i denotes the matrix obtained from A by removing row a^i, and where c^i and d^i denote the vectors obtained from c and d respectively by removing the i^{th} component. Let $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_{i-1}, \tilde{y}_{i+1}, \ldots, \tilde{y}_m)$ be an optimal integral solution of (3). Define $y^* = (\tilde{y}_1, \ldots, \tilde{y}_{i-1} \lceil \bar{y}_i \rceil, \tilde{y}_{i+1}, \ldots, \tilde{y}_m)$. Then y^* is integral and feasible to (2). We claim that y^* is in fact optimal to (2). To prove this claim, note that $(\bar{y}_1, \ldots, \bar{y}_{i-1}, \bar{y}_{i+1}, \ldots, \bar{y}_m)$ is feasible to (3). Therefore

$$\sum_{k \neq i} \tilde{y}_k \leq \sum_{k \neq i} \bar{y}_k.$$

In fact,

$$\sum_{k \neq i} \bar{y}_k - \sum_{k \neq i} \tilde{y}_k \geq \lceil \bar{y}_i \rceil - \bar{y}_i$$

because $\sum_{k \neq i} \bar{y}_k + \bar{y}_i$ is an integer by Lemma 4 and $\bar{y}_i$ is fractional. So

$$\sum_{k \neq i} \tilde{y}_k + \lceil \bar{y}_i \rceil \leq \sum_{k=1}^{m} \bar{y}_k,$$

i.e. y^* is an optimal integral solution to (2), and so the lemma must hold. ∎

Proof of Theorem 2. The necessity of the condition is immediate. We prove the sufficiency. Let A be the node-edge incidence matrix of a balanced hypergraph H. Then by Lemma 4, H has no perfect matching if and only if the objective value of the linear program

$$\begin{aligned} &\max\ 0x - \mathbf{1}s - \mathbf{1}t \\ (4) \qquad &Ax + Is - It = \mathbf{1} \\ &x,\ s,\ t \geq 0 \end{aligned}$$

is strictly negative. By Lemma 5, there exists an integral vector y such that

$$\begin{aligned} &y\mathbf{1} < 0 \\ (5) \qquad &yA \geq 0 \\ &-\mathbf{1} \leq y \leq \mathbf{1}. \end{aligned}$$

Set B to be the set of nodes i such that $y_i = -1$, R the set of nodes such that $y_i = 1$.

Then $yA \geq 0$ implies that each edge of H contains at least as many nodes in R as in B, and $y\mathbf{1} < 0$ implies $|B| > |R|$. ∎

Total dual integrality of the linear system $Ax + Is - It = \mathbf{1}$, x, s, $t \geq 0$ seems to be crucial to establish Hall's condition. For example, consider the perfect matrix below:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The system $Ax \leq \mathbf{1}$, $x \geq 0$ is TDI but not the system $Ax + Is - It = \mathbf{1}$ x, s, $t \geq 0$. The associated hypergraph H has no perfect matching, but this cannot be established with a bicoloring R, B satisfying the conditions of Theorem 2.

Standard proofs of Hall's theorem are combinatorial and quite easy. It would be nice to find a combinatorial proof of Theorem 2 which does not use the theory of integral polyhedra. For example, Hall's theorem can be easily proven from König's theorem. König's theorem also holds for balanced hypergraphs (Berge-Las Vergnas [3]). We do not know of a proof of Theorem 2 based on the Berge-Las Vergnas theorem. It would also be interesting to know if this theorem holds for a class of hypergraphs that strictly includes balanced hypergraphs.

Given a $0\ \pm 1$ matrix A, let $n(A)$ be the vector whose i^{th} component is the number of -1's in the i^{th} row of A. A $0\ \pm 1$ matrix A is *balanced* if for every square submatrix with two nonzeroes per row and column, the sum of the entries is congruent to 0 modulo 4. (If A is a 0, 1 matrix, this definition coincides with the previous one). Using the proof techniques in [4], it is possible to extend Lemma 5 to the following:

Lemma 6. *The linear system $Ax + Is - It = \mathbf{1} - n(A)$, $0 \leq x \leq \mathbf{1}$, s, $t \geq 0$ as TDI when A is a balanced $0, \pm 1$ matrix.*

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